Exercise 60

Find equations of the tangent line and normal line to the curve at the given point.

$$x^2 + 4xy + y^2 = 13, \quad (2, 1)$$

Solution

Take the derivative of both sides with respect to x.

$$\frac{d}{dx}(x^2 + 4xy + y^2) = \frac{d}{dx}(13)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(4xy) + \frac{d}{dx}(y^2) = 0$$

$$2x + \left[\frac{d}{dx}(4x)\right] y + 4x\left[\frac{d}{dx}(y)\right] + 2y \cdot \frac{dy}{dx} = 0$$

$$2x + (4)y + 4x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

Solve for dy/dx.

$$4x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = -2x - 4y$$

$$(4x + 2y)\frac{dy}{dx} = -2x - 4y$$

$$\frac{dy}{dx} = \frac{-2x - 4y}{4x + 2y}$$

$$= -\frac{x + 2y}{2x + y}$$

Plug in $x = 2$ and $y = 1$ to find the slope of the tangent line at the given point $(2, 1)$. The slope of the normal line is the negative reciprocal.

$$m_{\parallel} = -\frac{2 + 2(1)}{2(2) + 1} = -\frac{4}{5} \quad \Rightarrow \quad m_{\perp} = \frac{5}{4}$$

Use the point-slope formula with these slopes and the given point $(2, 1)$ to get the equations of the tangent and normal lines.

$$y - 1 = -\frac{4}{5}(x - 2) \qquad\qquad y - 1 = \frac{5}{4}(x - 2)$$

$$y - 1 = -\frac{4}{5}x + \frac{8}{5} \qquad\qquad y - 1 = \frac{5}{4}x - \frac{5}{2}$$

$$y = -\frac{4}{5}x + \frac{13}{5} \qquad\qquad y = \frac{5}{4}x - \frac{3}{2}$$

Below is a graph of the curve and its tangent and normal lines at $(2, 1)$.